Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Time 1 hour 30 minutes

Paper reference **9FM0/3A**

Further Mathematics

Advanced

PAPER 3A: Further Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- Good luck with your examination.

Turn over ▶

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1. The ellipse E has equation

$$\frac{x^2}{36} + \frac{y^2}{20} = 1$$

Find

(a) the coordinates of the foci of E,

(3)

(b) the equations of the directrices of E.

(2)

(a) $a^2 = 36 \Rightarrow a = 6$

from formula booklet:

given: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$b^2 = 20 \Rightarrow b = 2\sqrt{5}$

then: $b^2 = a^2(1-e^2)$

foci: $(\pm ae, 0)$

$b^2 = a^2(1-e^2)$

$20 = 36(1-e^2)$ ①

$1 - \frac{20}{36} = e^2 \Rightarrow e = \frac{2}{3}$ ①

$\therefore$ foci are $\pm 6 \times \frac{2}{3} \Rightarrow (\pm 4, 0)$ ①

(b) $x = \pm \frac{6}{2/3}$ ① $\leftarrow$ directrices $x = \pm \frac{a}{e}$

$x = \pm 9$ ①

Question 1 continued

(Total for Question 1 is 5 marks)

2. (i) Use the substitution $t = \tan\dfrac{x}{2}$ to prove the identity

$$\frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} \equiv \sec x + \tan x \qquad x \neq \frac{n\pi}{2} \qquad n \in \mathbb{Z}$$

(5)

(ii) Use the substitution $t = \tan\dfrac{\theta}{2}$ to determine the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{5}{4 + 2\cos\theta}\, d\theta$$

giving your answer in simplest form.

(5)

(i) $t = \tan\frac{x}{2}$

t-formulae:

$\sin x = \dfrac{2t}{1+t^2}$

$\cos x = \dfrac{1-t^2}{1+t^2}$

$$\frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} = \frac{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} + 1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} - 1} \quad (1)$$

$$= \frac{\left[\frac{2t - (1-t^2) + 1 + t^2}{1+t^2}\right]}{\left[\frac{2t + (1-t^2) - (1+t^2)}{1+t^2}\right]} \quad (1)$$

$1 + t^2$ cancels

$$= \frac{2t - (1-t^2) + 1 + t^2}{2t + (1-t^2) - (1+t^2)}$$

$$= \frac{2t^2 + 2t}{2t - 2t^2}$$

$\div 2t$

$$= \frac{t+1}{1-t} \quad (1)$$

$$= \frac{t+1}{1-t} \times \frac{1+t}{1+t}$$

$\leftarrow$ to rationalise

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Question 2 continued

$$= \frac{(t+1)(t+1)}{(1-t)(1+t)}$$

$$= \frac{t^2+2t+1}{1-t^2}$$

$$= \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2} \quad (1)$$

$\tan x = \frac{2t}{1-t^2}$

$$= \frac{1}{\cos x} + \tan x$$

$\sec x = \frac{1}{\cos x}$

$$= \sec x + \tan x \quad (1)$$

(ii) $t = \tan\frac{\theta}{2}$

$$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$$

$\sec^2\theta = 1 + \tan^2\theta$

$$dt = \frac{1+t^2}{2}\,d\theta$$

$$d\theta = \frac{2}{1+t^2}\,dt$$

when $\theta = \frac{\pi}{2}$,

$t = \tan\left(\frac{\frac{\pi}{2}}{2}\right) = 1$

$$\int_0^{\frac{\pi}{2}} \frac{5}{4+2\cos\theta}\,d\theta = \int_0^1 \frac{5}{4+2\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}\,dt \quad (1)$$

$$= \int_0^1 \frac{10}{4(1+t^2)+2(1-t^2)}\,dt$$

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Question 2 continued

$$= \int_0^1 \frac{10}{4 + 4t^2 + 2 - 2t^2} \, dt$$

$$= \int_0^1 \frac{5}{3 + t^2} \, dt \quad 1$$

recognise from formula book

$$= \left[5 \times \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) \right]_0^1 \quad 1 \qquad \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$= \left[5 \times \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \right] - \left[5 \times \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{0}{\sqrt{3}}\right) \right] \quad 1$$

$$= \frac{5}{\sqrt{3}} \times \frac{\pi}{6} - 0$$

$$= \frac{5\pi}{6\sqrt{3}} \quad 1$$

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Question 2 continued

(Total for Question 2 is 10 marks)

3.

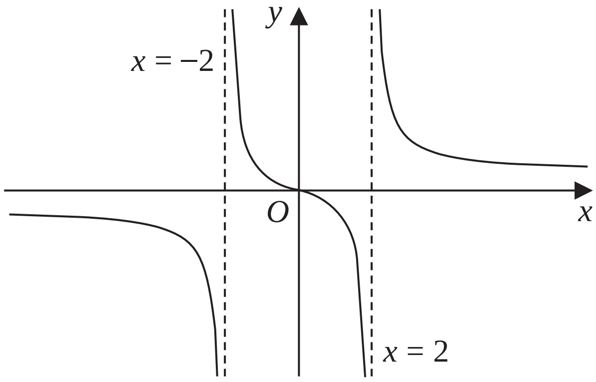

Figure 1

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{x}{|x| - 2}$$

Use algebra to determine the values of x for which

$$2x - 5 > \frac{x}{|x| - 2}$$

(8)

When $x \geqslant 0$, $|x| = x$ so $2x - 5 > \dfrac{x}{x - 2}$. 1

Critical value/s : $(2x-5)(x-2) = x$ 1 ← we know $x \neq 2$ so we don't need to square the denominator

$2x^2 - 9x - x + 10 = 0$

$2x^2 - 10x + 10 = 0$

$x = \dfrac{-10 \pm \sqrt{(-10)^2 - 2 \times (-10) \times 10}}{2 \times 2}$

$x = \dfrac{5 \pm \sqrt{5}}{2}$ 1

When $x < 0$ $-|x| = x$ so $2x - 5 > \dfrac{x}{-x - 2}$ 1

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Question 3 continued

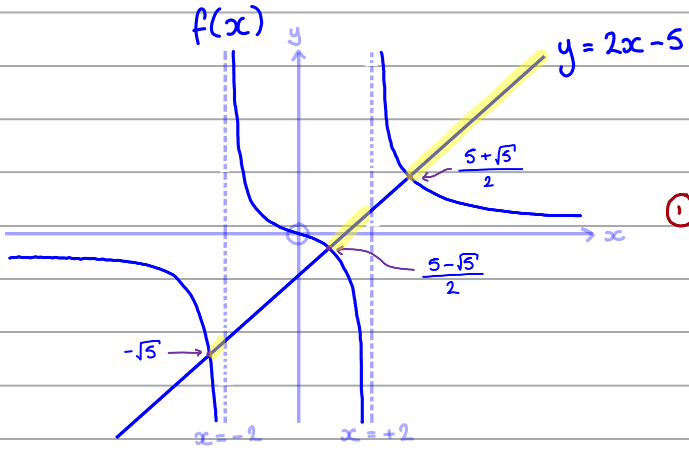
Critical value/s : $(-x-2)(2x-5) = x$

$-2x^2 - 4x + 5x - x + 10 = 0$

$10 - 2x^2 = 0$

$5 = x^2$

$x = +\sqrt{5}$ is invalid because we are considering when $x < 0$ → $-\sqrt{5} = x$ (1)



(1)

The line $y = 2x - 5$ is higher than $y = \dfrac{x}{|x|-2}$ when:

$-\sqrt{5} < x < -2$ (1) and $\dfrac{5-\sqrt{5}}{2} < x < 2$ and $x > \dfrac{5+\sqrt{5}}{2}$ (1)

Question 3 continued

Question 3 continued

(Total for Question 3 is 8 marks)

4.

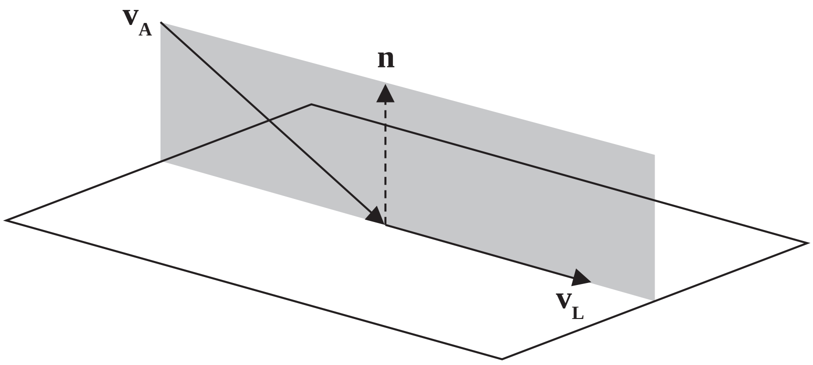


Figure 2

A small aircraft is landing in a field.

In a model for the landing the aircraft travels in different straight lines before and after it lands, as shown in Figure 2.

The vector $\mathbf{v}_A$ is in the direction of travel of the aircraft as it approaches the field.

The vector $\mathbf{v}_L$ is in the direction of travel of the aircraft after it lands.

With respect to a fixed origin, the field is modelled as the plane with equation

$$x - 2y + 25z = 0$$

and

$$\mathbf{v}_A = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

(a) Write down a vector $\mathbf{n}$ that is a normal vector to the field.

(1)

(b) Show that $\mathbf{n} \times \mathbf{v}_A = \lambda \begin{pmatrix} 13 \\ 19 \\ 1 \end{pmatrix}$, where λ is a constant to be determined.

(2)

When the aircraft lands it remains in contact with the field and travels in the direction $\mathbf{v}_L$

The vector $\mathbf{v}_L$ is in the same plane as both $\mathbf{v}_A$ and $\mathbf{n}$ as shown in Figure 2.

(c) Determine a vector which has the same direction as $\mathbf{v}_L$

(3)

(d) State a limitation of the model.

(1)

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Question 4 continued

(a) $n = \begin{pmatrix} 1 \\ -2 \\ 25 \end{pmatrix} \leftarrow n_1 x + n_2 y + n_3 z = d$

and $n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

(b) $n \times V_a = \begin{vmatrix} i & j & k \\ 1 & -2 & 25 \\ 3 & -2 & -1 \end{vmatrix} \begin{matrix} \\ \leftarrow n \\ \leftarrow V_a \end{matrix}$

$= i(-2\times-1 - 25\times-2) - j(1\times-1 - 25\times3) + k(1\times-2 - (-2\times3))$

$= 52i + 76j + 4k$

$= 4\begin{pmatrix} 13 \\ 19 \\ 1 \end{pmatrix}$ where $\lambda = 4$

(c) V_L in same plane as (and perpendicular to) $n \times V_A$ and n:

$d = (n \times V_A) \times n$

$d = \begin{bmatrix} 13 \\ 19 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ 25 \end{bmatrix} \leftarrow n$

direction of $n \times V_A$

$= \begin{vmatrix} i & j & k \\ 13 & 19 & 1 \\ 1 & -2 & 25 \end{vmatrix}$

$= (19\times25 - 1\times-2)i - (13\times25 - 1\times1)j + (13\times-2 - 1\times19)k$

$= \begin{bmatrix} 477 \\ -324 \\ -45 \end{bmatrix}$

Question 4 continued

(d) There may be some lateral movement

↑

side-to-side, not flat
on the plane / line, etc

Question 4 continued

(Total for Question 4 is 7 marks)

5. The parabola C has equation

$$y^2 = 32x$$

and the hyperbola H has equation

$$\frac{x^2}{36} - \frac{y^2}{9} = 1$$

(a) Write down the equations of the asymptotes of H.

(1)

The line l_1 is normal to C and parallel to the asymptote of H with positive gradient.

The line l_2 is normal to C and parallel to the asymptote of H with negative gradient.

(b) Determine

(i) an equation for l_1

(ii) an equation for l_2

(4)

The lines l_1 and l_2 meet H at the points P and Q respectively.

(c) Find the area of the triangle OPQ, where O is the origin.

(4)

(a) For H. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, asymptotes are: $\frac{x}{a} = \pm \frac{y}{b}$

$a^2 = 36 \Rightarrow a = 6$, $b^2 = 9 \Rightarrow b = 3$

$\frac{x}{6} = \pm \frac{y}{3} \Rightarrow \therefore y = \pm \frac{1}{2}x$ 1

(b) $y^2 = 32x$

$2y\frac{dy}{dx} = 32$

gradient of normal $m_N \times m_T = -1$
so differentiate C for gradient.

$\frac{dy}{dx} = \frac{16}{y}$ 1

$m_N = \frac{-1}{\frac{16}{y}} = -\frac{y}{16}$

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Question 5 continued

$y = \pm \frac{1}{2}x$ $\quad \frac{dy}{dx} = \pm \frac{1}{2}$ ← l_1 and l_2 are parallel to asymptotes

$-\frac{y}{16} = \pm \frac{1}{2}$ $\quad \therefore m_N = \frac{dy}{dx}$

$y = \pm \frac{16}{2} = \pm 8$

Using $y^2 = 32x$, $x = \frac{8^2}{32} = 2$ 1

Points of contact are $(8, 2)$ and $(-8, 2)$.

To find l_1: $\quad y-(-8) = \frac{1}{2}(x-2)$ ← using +ve gradient

$y = \frac{1}{2}x - 9$ 1

To find l_2: $\quad y - 8 = -\frac{1}{2}(x-2)$ ← using −ve gradient

$y = -\frac{1}{2}x + 9$ 1

(c) Where lines meet H: $\quad \frac{x^2}{36} - \frac{\left(\pm\left(\frac{1}{2}x - 9\right)\right)^2}{9} = 1$

equation of H

× 36

$x^2 - 4\left(\frac{1}{2}x - 9\right)^2 = 36$

$x^2 - 4\left(\frac{1}{4}x^2 - 9x + 81\right) = 36$

$36x = 360$

$x = 10$ 1

Question 5 continued

to find y $\rightarrow$ $\frac{x^2}{36} - \frac{y^2}{9} = 1$ $\times 36$

$100 - 4y^2 = 36$

$64 = 4y^2$

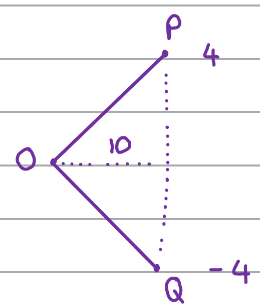
$16 = y^2$

$y = \pm 4$

$\therefore P = (10, \pm 4)$ 1

Area OPQ $= \frac{1}{2} \times 10 \times (4 - (-4))$ 1

$= 40$ 1



Question 5 continued

(Total for Question 5 is 9 marks)

6.

$$\left[\begin{array}{c}\textit{The Taylor series expansion of } \mathrm{f}(x) \textit{ about } x = a \textit{ is given by} \\ \mathrm{f}(x) = \mathrm{f}(a) + (x-a)\mathrm{f}'(a) + \dfrac{(x-a)^2}{2!}\mathrm{f}''(a) + \ldots + \dfrac{(x-a)^r}{r!}\mathrm{f}^{(r)}(a) + \ldots\end{array}\right]$$

Given that

$$y = (1 + \ln x)^2 \qquad x > 0$$

(a) show that $\dfrac{d^2y}{dx^2} = -\dfrac{2\ln x}{x^2}$

(4)

(b) Hence find $\dfrac{d^3y}{dx^3}$

(2)

(c) Determine the Taylor series expansion about $x = 1$ of

$$(1 + \ln x)^2$$

in ascending powers of $(x - 1)$, up to and including the term in $(x - 1)^3$

Give each coefficient in simplest form.

(3)

(d) Use this series expansion to evaluate

$$\lim_{x \to 1} \frac{2x - 1 - (1 + \ln x)^2}{(x - 1)^3}$$

explaining your reasoning clearly.

(3)

(a) $y = (1 + \ln x)^2$

$\dfrac{dy}{dx} = 2 \times \left(\dfrac{1}{x}\right) \times (1 + \ln x)^{2-1}$ 1

$\dfrac{dy}{dx} = \dfrac{2}{x}(1 + \ln x)$ 1

Using product rule:

$u = \dfrac{2}{x}$ $\qquad v = (1 + \ln x)$

$u' = -\dfrac{2}{x^2}$ $\qquad v' = \dfrac{1}{x}$

P 6 6 7 9 8 R A 0 2 0 3 2

Question 6 continued

$$uv' + vu' = \frac{2}{x} \times \frac{1}{x} + \frac{-2}{x^2} \times (1 + \ln x) \quad 1$$

$$\therefore \frac{d^2y}{dx^2} = \frac{2 - 2(1 + \ln x)}{x^2}$$

$$\frac{d^2y}{dx^2} = -\frac{2\ln x}{x^2} \quad 1$$

(b) Using product rule:

$$u = -2\ln x \qquad v = x^{-2}$$

$$u' = -\frac{2}{x} \qquad v' = -2x^{-3}$$

$$uv' + vu' = -2\ln x \times -2x^{-3} + x^{-2} \times -\frac{2}{x} \quad 1$$

$$\therefore \frac{d^3y}{dx^3} = \frac{4\ln x}{x^3} - \frac{2}{x^3} \quad 1$$

(c) When $x = 1$: $y(1) = (1 + \ln 1)^2 = 1$

$$y'(1) = \frac{2}{1}(1 + \ln(1)) = 2$$

$$y''(1) = \frac{-2\ln(1)}{1^2} = 0$$

$$y'''(1) = \frac{4\ln(1)}{1^3} - \frac{2}{1^3} = -2 \quad 1$$

$$y = 1 + 2(x-1) + \frac{0}{2!}(x-1)^2 + \frac{-2}{3!}(x-1)^3 + \ldots \quad 1$$

Question 6 continued

$$y = 1 + 2(x-1) - \frac{1}{3}(x-1)^3 + \ldots \quad (1)$$

(d) Substitute $(1+\ln x)^2 \rightarrow y = 1 + 2(x-1) \ldots$

$$\frac{2x-1-(1+\ln x)^2}{(x-1)^3} = \frac{2x-1-\left[1+2(x-1)-\frac{1}{3}(x-1)^3\right]}{(x-1)^3} \quad (1)$$

$$= \frac{2x-2-2x+2 + \frac{1}{3}(x-1)^3}{(x-1)^3}$$

$$= \frac{\frac{1}{3}\cancel{(x-1)^3}}{\cancel{(x-1)^3}} \quad (1)$$

$\therefore$ Hence $\lim_{x \to 1} \frac{2x-1-(1+\ln x)^2}{(x-1)^3} = \frac{1}{3}$ because all remaining terms will become zero in the limit as they are multiples of $(x-1)^k$. (1)

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Question 6 continued

(Total for Question 6 is 12 marks)

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7. With respect to a fixed origin O, the line l has equation

$$(\mathbf{r} - (12\mathbf{i} + 16\mathbf{j} - 8\mathbf{k})) \times (9\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}) = \mathbf{0}$$

position vector of any point on the line ↗ ↑ point on the line ↑ a vector parallel to the line

The point A lies on l such that the direction cosines of $\overrightarrow{OA}$ with respect to the **i**, **j** and **k** axes are $\frac{3}{7}$, β and γ.

Determine the coordinates of the point A.

(7)

$\overrightarrow{OA}$ is given by $r = a + \lambda d$.

$\overrightarrow{OA} = (12i + 16j - 8k) + \lambda(9i + 6j + 2k)$

$\overrightarrow{OA} = (12 + 9\lambda)i + (16 + 6\lambda)j + (-8 + 2\lambda)k$ (1)

$\cos\alpha = \frac{x}{|a|}$ where $a = xi + yj + zk$

$\frac{3}{7} = \frac{12 + 9\lambda}{\sqrt{(12+9\lambda)^2 + (16+6\lambda)^2 + (-8+2\lambda)^2}}$ (1)

$3\sqrt{144 + 216\lambda + 81\lambda^2 + 256 + 192\lambda + 36\lambda^2 + 64 - 32\lambda + 4\lambda^2} = 7(12 + 9\lambda)$

$3^2(464 + 376\lambda + 121\lambda^2) = (84 + 63\lambda)^2$ (1)

$4176 + 3384\lambda + 1089\lambda^2 = 7056 + 10584\lambda + 3969\lambda^2$

$2880\lambda^2 + 7200\lambda + 2880 = 0$

$\div 1440$

$2\lambda^2 + 5\lambda + 2 = 0$ (1)

$(2\lambda + 1)(\lambda + 2) = 0$

$\lambda = -\frac{1}{2}, -2$ (1)

substituting $\lambda = -2$ gives
$\overrightarrow{OA} = -6i + 4j - 12k$.
$\cos\alpha = \frac{-6}{\sqrt{196}} = -\frac{3}{7}$ which doesn't match the given value

$\overrightarrow{OA} = \begin{bmatrix} 12 + 9(-\frac{1}{2}) \\ 16 + 6(-\frac{1}{2}) \\ -8 + 2(-\frac{1}{2}) \end{bmatrix}$ (1) $= \begin{bmatrix} \frac{15}{2} \\ 13 \\ -9 \end{bmatrix}$

$\therefore A = \left(\frac{15}{2}, 13, -9\right)$ (1)

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Question 7 continued

Question 7 continued

Question 7 continued

(Total for Question 7 is 7 marks)

8. A community is concerned about the rising level of pollutant in its local pond and applies a chemical treatment to stop the increase of pollutant.

The concentration, x parts per million (ppm), of the pollutant in the pond water t days after the chemical treatment was applied, is modelled by the differential equation

$$\frac{dx}{dt} = \frac{3 + \cosh t}{3x^2 \cosh t} - \frac{1}{3} x \tanh t \qquad \text{(I)}$$

When the chemical treatment was applied the concentration of pollutant was 3 ppm.

(a) Use the iteration formula

$$\left(\frac{dy}{dx}\right)_n \approx \frac{(y_{n+1} - y_n)}{h}$$

once to estimate the concentration of the pollutant in the pond water 6 hours after the chemical treatment was applied.

(4)

(b) Show that the transformation $u = x^3$ transforms the differential equation (I) into the differential equation

$$\frac{du}{dt} + u \tanh t = 1 + \frac{3}{\cosh t} \qquad \text{(II)}$$

(3)

(c) Determine the general solution of equation (II)

(4)

(d) Hence find an equation for the concentration of pollutant in the pond water t days after the chemical treatment was applied.

(3)

(e) Find the percentage error of the estimate found in part (a) compared to the value predicted by the model, stating if it is an overestimate or an underestimate.

(3)

6 hours $= \frac{6}{24}$ (0.25) days so $h = 0.25$ 1

when $t = 0$, $x = 3$.

$$\frac{dx}{dt} = \frac{3 + \cosh 0}{3 \times 3^2 \cosh 0} - \frac{1}{3}(3)\tanh 0$$

$\cosh 0 = \frac{e^0 + e^{-0}}{2} = 1$

$$\frac{dx}{dt} = \frac{4}{27}$$

$\tanh 0 = \frac{e^{2\times 0} - 1}{e^{2\times 0} + 1} = 0$

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Question 8 continued

$$\left(\frac{dx}{dt}\right)_n \approx \frac{x_{n+1} - x_n}{h} \Rightarrow x_{n+1} \approx h\left(\frac{dx}{dt}\right)_n + x_n$$

$$x_1 \approx 0.25 \times \frac{4}{27} + 3 = \frac{82}{27} \quad 1$$

$\therefore$ After 6 hours, concentration of the pollutant is approximately 3.04 ppm (3 s f) 1

(b) $$\frac{dx}{dt} = \frac{3 + \cosh t}{3x^2 \cosh t} - \frac{1}{3} x \tanh t$$

$$\frac{du}{dt} = \frac{du}{\cancel{dx}} \times \frac{\cancel{dx}}{dt} \qquad u = x^3 \Rightarrow \frac{du}{dx} = 3x^2$$

$$\frac{du}{dt} = 3x^2 \times \left[\frac{3 + \cosh t}{3x^2 \cosh t} - \frac{1}{3} x \tanh t\right] \quad 1$$

$$\frac{du}{dt} = \frac{3 + \cosh t}{\cosh t} - \frac{3x^2}{3} x \tanh t$$

$$\frac{du}{dt} = \frac{3}{\cosh t} + \frac{\cosh t}{\cosh t} - x^3 \tanh t \qquad u = x^3$$

$$\frac{du}{dt} = \frac{3}{\cosh t} + 1 - u \tanh t \quad 1$$

$$\frac{du}{dt} + u \tanh t = \frac{3}{\cosh t} + 1 \quad 1$$

Question 8 continued

(c) $\frac{du}{dt} + u\tanh t = \frac{3}{\cosh t} + 1$

This is in the form $\frac{dy}{dx} + P(x)y = Q(x)$ so we can use the integrating factor (IF) method with the

$IF = e^{\int P(x)\,dx}$

$IF = e^{\int \tanh t \, dt}$ $\int \tanh x \, dx = \ln(\cosh x)$ from formula book

$IF = e^{\ln(\cosh t)}$ $e^{\ln x} = x$

$IF = \cosh t$ (1)

Multiply through by the integrating factor:

$\cosh t \frac{du}{dt} + u\tanh t\cosh t = \cosh t\left(\frac{3}{\cosh t} + 1\right)$

$\cosh t \frac{du}{dt} + u\sinh t = \cosh t + 3$

Notice that $\frac{d}{dt}(\cosh t) = u\sinh t$, so the $\sinh t$ term can be 'absorbed' into the differential term.

$\frac{d}{dt} u\cosh t = \cosh t + 3$

$\int \frac{d}{dt} u\cosh t \, dt = \int \cosh t + 3 \, dt$ (1) $\int \cosh t = \sinh t$

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Question 8 continued

$$u\cosh t = \sinh t + 3t + c \quad (1)$$

rearrange for $u =$

$$u = \frac{\sinh t + 3t + c}{\cosh t}$$

$$u = \tanh t + \frac{3t}{\cosh t} + \frac{c}{\cosh t} \quad (1)$$

(d) when $t = 0$, $x = 3$ and $u = 3^3 = 27$

$$27 = \tanh 0 + \frac{3(0)}{\cosh 0} + \frac{c}{\cosh 0}$$

$\tanh 0 = 0$
$\cosh 0 = 1$

$$27 = 0 + 0 + c$$

$$c = 27 \quad (1)$$

$$x = \left(\tanh t + \frac{3t + 27}{\cosh t}\right)^{\frac{1}{3}} \quad (2)$$

← using $u = x^3$

(e) Using the model :

$$x(0.25) = \left(\tanh(0.25) + \frac{0.75 + 27}{\cosh(0.25)}\right)^{\frac{1}{3}}$$

$$= 3.0055.. \quad (1)$$

estimate from part (a)

$$\%\ \text{error} = \frac{3.0055.. - \frac{82}{27}}{3.0055...} = -1.05\,\% \quad (1)$$

∴ Estimate in (a) is an overestimate by 1.05%. (1)

Question 8 continued

(Total for Question 8 is 17 marks)

TOTAL FOR PAPER IS 75 MARKS

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